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Volume optimization of end-clamped arches

Optimización de volumen para arcos empotrados

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ABSTRACT

Even if arch arised as structural system more than two thousand years ago, this structural typology is still not widely diffused and is mainly adopted when large spans have to be covered. The structural efficiency of arches primarily depends on optimal material exploitation, i.e. minimization of internal stress eccentricity that reduces structural material volume and weight. An efficient structure, under these terms, implies simple and light scaffolding, so contributing in minimizing construction costs.

Although very abundant knowledge and literature on arches, there is still scope for design optimization. This study is framed within this context and deals with the structural analysis of end-clamped plane circular arches under uniformly distributed vertical load and self weight. In the first step, the analytical solution of arch static and kinematic behaviour is derived by the force method. In the second step, the arch shape is optimized, by assuming the arch volume, and thus the weight, as objective function. Finally minima of the objective function (i.e. optimal geometric shape parameters) are computed and charted in order to be used for practical purposes.

KEYWORDS: Arch; static behaviour; force method; optimization; arch volume; objetive function.

RESUMEN

Aunque el arco surgió como sistema estructural hace más de dos mil años, esta tipología estructural todavía no está muy difundida y se adopta principalmente cuando hay que cubrir grandes luces. La eficiencia de los arcos depende principalmente de la explotación óptima del material, es decir, de la minimización de la excentricidad del estrés, que reduce el volumen y el peso del material estructural. Una estructura eficiente, en estos términos, implica andamios simples y ligeros, contribuyendo así a minimizar los costes de construcción.

Aunque hay muchos conocimientos y literatura sobre arcos, todavía hay margen para la optimización del diseño. El presente estudio se enmarca en este contexto y se ocupa del análisis estructural de los arcos circulares planos empotrados bajo una carga vertical distribuida uniformemente y un peso propio. En el primer paso, la solución analítica del comportamiento estático y cinemático de los arcos se estudia por el método de la fuerza. En el segundo paso, se optimiza la forma del arco, asumiendo el volumen del arco, y por lo tanto el peso, como función objetiva. Finalmente se calculan los mínimos de la función objetiva (es decir, los parámetros óptimos de la forma geométrica) para poder utilizarlos con fines prácticos..

PALABRAS CLAVE: Arco; comportamiento estático; método de fuerza; optimización; volumen del arco; función objetiva.

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1.
INTRODUCCIÓN

Arches are inherently efficient structures; they are capable to transfer loads from the superstructure to the foundations [1] with low structural weight. If properly shaped, they become the optimal solution to cross large spans and transfer high loads. Structural efficiency depends on the predominance of centered normal stress [2-4]: in this circumstance smaller cross sections can be used with respect to beams. Contrarily internal stress eccentricity (i.e. large bending moments) or large shear stress should be avoided, leading to uneconomical design, sub-exploitation of building materials and unnecessary self weight [5,6]. Further design economy can be achieved via overall shape and cross section optimization, aimed at satisfying specific objectives and constraints. A key point is in many cases the minimization of structural volume, since arch self weight is the largest component of the vertical load, accounting for about half the total.

Optimization is a key issue for good design. From the data of 55 arch bridges built during the twentieth century reported in [7] several empirical lessons may be learnt. The first one is that (long span) concrete arches consume, per unit length, higher material quantities as compared to (shorter span) post tensioned concrete girder bridges. This is an expected result, at least since arches are curved, whereas beams are not; however, post tensioned concrete girder are not usable on large spans. The second lesson is that, for long span arch bridges, arch self weight is about half the total vertical load.

Both lessons further motivate the search for optimal (less material consuming) solutions. Further, structural optimization is an important design tool for shape selection, also from an architectural view point.

Structural optimization has been common for a long time in mechanical and aeronautical engineering. In civil engineering, it has been progressively adopted more recently, for both buildings and bridges [8-11].

Traditionally, it is since the seventeenth century that firstly Galileo and then Hooke approached the hanged chain problem, but more accurate solutions, published on Acta Eruditorum, are due to Bernoulli, Leibniz and Huygens. Since then, this shape has been addressed as optimal solution for compressive arch ribs under directly applied loads, or for suspended cables in tension. Catenary arches show properties of pure compression, without bending moment or shear stress. A chain suspended between two points will form this unique curve, which is routinely used for arches, and sometimes for shells (although this is not fully correct due to bi-dimensional stiffness).

It is worth to remember that Hooke, as reported by Heyman, was the first experimentalist; he introduced the concept of inverted catenary as optimal arch form. Significant support was also given by Gaudi, Otto and Isler during the nineteenth and twentieth century.

A more analytical study was the one from Ramsey that in 1953 at Cambridge derived geometrical configurations of flexible chains and strings, comparable to optimal arch design. More recent studies, carried out by Rozvany and Prager [12] were focused on searching for the optimal volume by minimizing the angle between the parabolic funicular and the vertical axis at support. The prerequisite of zero tensile strength is a common assumption to demonstrate the parabolic funicular shape, implying the absence of bending moment. It was proved that the parabolic funicular is optimal when tensile stress does not exceed one-third of compressive stress. A numerical approach employing this limiting conditions was presented by Darwich et al [13].

Thanks to a large-scale layout optimization technique developed by Gilbert and Tyas [14], it has been proved that an optimal structural performance can be obtained by adopting truss structures connecting the supports to the end points of a central parabolic section.

Another very recent analytical study about arch configuration is due to Osserman [15]; he specifies in a precise and mathematical fashion the confusion on the Gatway Arch shape in St. Louis.

A challenging view on these results can be found in Tyas et al [16] where it is proved, by numerical evidences, that a parabolic funicular is not necessarily the optimal structural form to carry a uniform load between fixed supports; so an explicit analytical expression for geometry and stress is proposed in order to design suitable truss systems emerging from the supports and thus obtain a global optimization.

A fresh look upon optimization approach is also presented in the study from Vanderplaats and Han [17], where an optimization technique based on an iterative force approximation method is combined with a finite-element technique to obtain a minimum arch volume, by assuming variable cross-section and simply supported or fixed end-constrains.

A very interesting study on moment-less arches is finally proposed by Lewis [18]. In his mathema-tical model, a prediction on a simply supported arch rib shape is presented. Both arch selfweight and a uniformly distributed load are included in the analysis in order to show which geometry, among pa-rabolic or catenary arch, is the most suitable one. Results show that catenary arch shape produces lower stresses.

2.
PROBLEM STATEMENT

2.1. Geometry

A geometrical description of a curved beam can be given through a 1-D solid with a centroid curve Γ and with a cross section A associated at each point of Γ (figure 1 left). It is assumed that the plane of Γ is also a plane of mechanical symmetry. In figure 1 (right) the geometric scheme of the right half

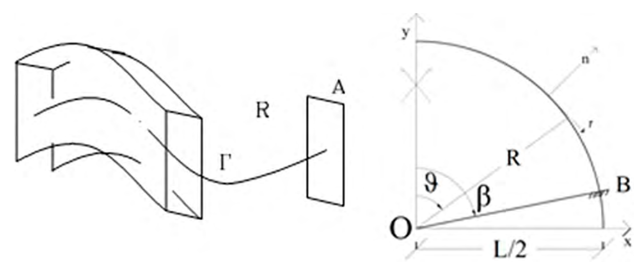

Figure 1. Geometry (left); assumptions (right).

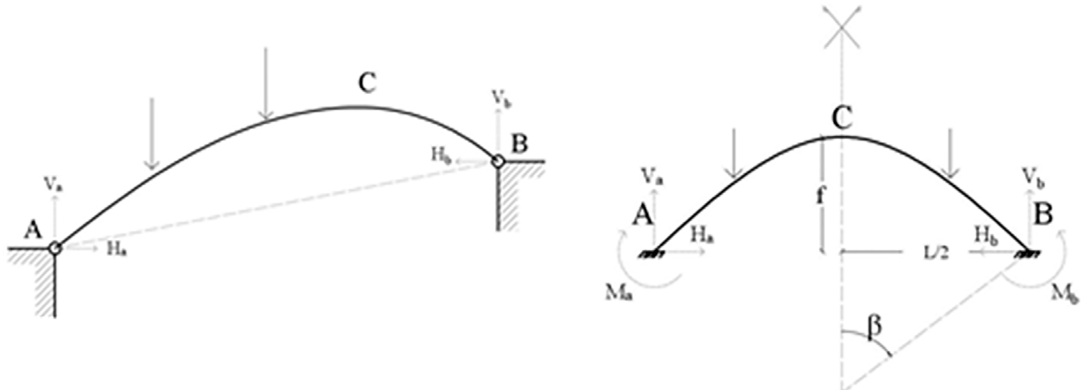

Figure 2. Two-hinged arch (left); clamped arch (right).

of the arch is represented, in which ϑ is the colatitude of the generic section and β the colatitude of the end section.

The following images show two generic problems of curved beams with different boundary conditions and different curvatures. Figure 2 (left) shows an hinged arch, asymmetrically loaded with two external forces in a generic position and with supports at different heights. Figure 2 (right) shows a parabolic symmetrical clamped arch, loaded with two external forces. Our interest will be focused on clamped circular arches, symmetrically loaded.

2.2. Constitutive bond and Kinematics

In this study, we assume that the only non null deformation is the curvature κ, given by:

$$M = EJ\kappa \tag{1}$$

Moreover displacements and deformations are assumed small.

2.3. Loads

The arch is subjected to its self-weight and to a distributed external load for unit horizontal length $q(x)$ (figure 3 left). The arch is made up of a homogeneous material with specific gravity γ. So the tangent (p) τ and normal (p_n) projections of the resultant load are given by:

$$\begin{cases} p_\tau = (\frac{q}{2} + \gamma A)\sin\vartheta \\ p_n = (q\cos\vartheta - \gamma A)\cos\vartheta \end{cases} \tag{2}$$

The Force method is applied, that in this context assumes a very simple and effective form. In fact hyperstatic unknowns can be determined by solving the internal work integral including in it just the bending component.

The parametric variables inherent the arch geometry are: R, β and λ, the first two expressing the radius and the opening conditions of the semicircle, while the last one is a slenderness parameter. More precisely $\lambda = L/\overline{w}$, where $\overline{w} = W/A$ is the vertical semi-dimension of the section core, equal to the ratio of the section modulus W over the cross section area A.

Also the load ratio $\mu = \gamma A/q$ is introduced, expressing the ratio between the self weight and the uniformly distrubuted load.

The equilibrium equations, according to the adopted reference system, can be written as:

$$\begin{cases} N'(s) + \dfrac{T(s)}{R} = -q\left[\mu \sin\left(\dfrac{s}{R}\right) + \dfrac{1}{2}\sin\left(\dfrac{2s}{R}\right)\right] \\ T'(s) - \dfrac{N(s)}{R} = q\left[\mu\cos\left(\dfrac{s}{R}\right) + \cos^2\left(\dfrac{s}{R}\right)\right] \\ M'(s) + T'(s) = 0 \end{cases} \tag{3}$$

with the boundary conditions:

$$\begin{cases} T(0) = 0 \\ N(0) = -H \\ M(\beta) = -X \end{cases} \tag{4}$$

where s = ϑR, while X and H are the hyperstatic unknowns. The following variable substitution is then applied:

$$R = \frac{L}{2\sin(\beta)} \tag{5}$$

and the following dimensionless mechanical variables are considered:

$$\begin{aligned} &n = \frac{N}{qL}; \quad t = \frac{T}{qL}; \quad m = \frac{T}{qL^2}; \\ &h = \frac{H}{qL}; \quad x = \frac{X}{qL^2} \end{aligned} \tag{6}$$

From Eqs. (3) and (6) the dimensionless internal forces are obtained:

$$\begin{aligned} &n(\vartheta,\beta,h) = -\frac{1}{2}\csc(\beta)\sin(\vartheta)(\mu\vartheta + \sin(\vartheta))h\cos(\vartheta) \\ &t(\vartheta,\beta,h) = -\frac{1}{2}\csc(\beta)\cos(\vartheta)(\mu\vartheta + \sin(\vartheta)) - h\sin(\vartheta) \\ &m(\vartheta,\beta,h,x) = \frac{1}{8}(\csc^2(\beta)(-2\mu\vartheta\sin(\vartheta) - 2\mu\cos(\vartheta) + \\ &\cos^2(\vartheta)) - \cot^2(\beta) + 2\csc(\beta)(\beta\mu - 2h\cos(\vartheta)) + \\ &2\cot(\beta)(\mu\csc(\beta) + 2h) - 8x) \end{aligned} \tag{7}$$

Therefore, by means of virtual work theorem, the following kinematic conditions are imposed:

$$\begin{cases} u_x(B)=0 \\ \gamma(B)=0 \end{cases} \tag{8}$$

where $u_x(B)$ and $\gamma(B)=0$ are the horizontal displacement and the rotation at the end section B. The principal system, statically determined, is shown in figure 3 (right), where also the hyperstatic unknowns X and H are indicated.

The dimensionless hyperstatic unknowns x and h are thus determined by Eqs. (8):

$$x(B)=-csc(\beta)\left[\frac{-96\,\beta^3\mu\,sin(\beta)-36(4\beta^2+1)\mu cos(\beta)+8(-3\beta^2-4)cos\,(2\beta)}{192\,(2\beta^2+\beta\,sin(2\beta)+2\,cos(2\beta)-2}+\frac{252\,\beta\mu sin(\beta)+12\ \beta\mu sin(3\beta)}{192\,(2\beta^2+\beta\,sin(2\beta)+2\,cos(2\beta)-2)}\right.$$
$$\left.+\frac{36\mu cos(3\beta)-28\,sin(2\beta)+2\,sin(4\beta)+5\,cos\,(4\beta)+27}{192\,(2\beta^2+\beta\,sin(2\beta)+2\,cos(2\beta)-2)}\right] \tag{9}$$

$$h(\beta)=-csc(\beta)\left[\frac{-24\,\beta^2\mu+12\,(\beta^2-4)\mu cos(2\beta)-42\,\beta\mu sin(2\beta)}{24\,(2\beta^2\mu+\beta\,sin(2\beta)+2\,cos(2\beta)-2)}+\frac{6\beta\,sin(\beta)+2\beta\,sin(3\beta)-3cos(\beta)+3cos(3\beta)+48\mu}{24\,(2\beta^2+\beta\,sin(2\beta)+2\,cos(2\beta)-2)}\right.$$

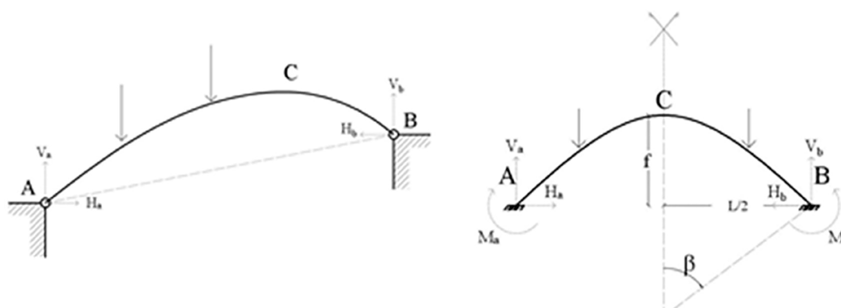

Figure 3. Load configuration (left); principal system for force method (right).

4. OPTIMAL SOLUTION

The arch that minimizes the volume is here determined, assuming the volume representative of the material cost.

The optimal solution is found through suitable considerations on dimensionless bending moment m values at clamped sections.

It is in fact assumed that the maximum stress is attained at the clamped sections. So, in order to solve the optimal problem, the stress at clamped sections under axial-bending condition is set equal to the limit stress. Then the following optimization condition is imposed:

$$\frac{M}{\overline{w}A}-\frac{N}{A}=f_d \tag{10}$$

that in dimensionless form can be rewritten as:

$$-n+\lambda m=\frac{f_d A}{qL} \tag{11}$$

Volume can be finally obtained deriving the cross section A from Eq. (11):

$$V=2AR\beta=\frac{AL\beta}{sin\beta}=(-n+\lambda m)\,\frac{qL^2}{f_d}\,\frac{\beta}{sin\beta} \tag{12}$$

Figure 4 shows the trend of function m versus the colatitude θ and the load ratio π, for $\pi=\beta/6$, restrict-ing the analysis in the ranges $0 < \vartheta < \beta$; $0 < \beta < \pi/2$, $0 < \mu < 10$. In figure 5 the same graphs are depicted in the plane m–θ.

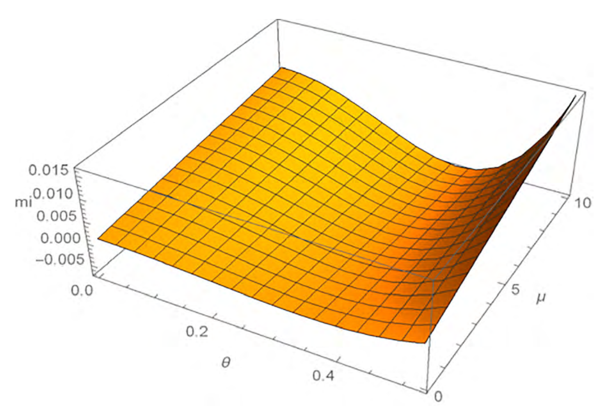

Figure 4. Function m versus θ and μ, for $\beta=\pi/6$: front view (left) and assonometric view (right).

From numerical analysis it emerges that, under our assumptions, the maximum stress is attained at clambed end-section,

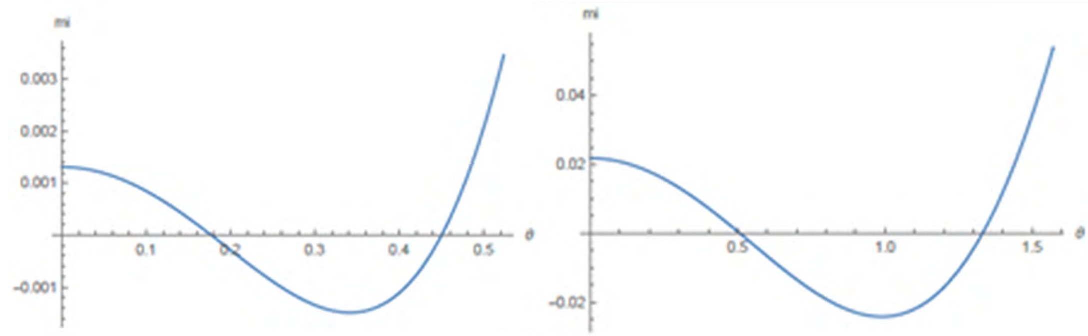

Figure 5. Function m versus θ and μ, for $\beta = \pi/6$ (left) and $\beta = \pi/2$ (right).

as previously supposed, so allowing to assume Eq. (10) as design constraint.

For design purposes, a new variable, the dimensionless span $\eta = L/\bar{h}$, is introduced, expressing the ratio between the arch span L and the height $\bar{h} = f_d/\gamma$ of a column made by the same material of the arch and subjected to its self-weight, in which the prescribed normal stress f_d is attained at the base section.

As above stated, by Eq. (12) it is possible to achieve the minimum arch volume able to carry self weight and applied load, in which the maximum normal stresses at the end-section do not exceed the limit value f_d. A logarithmic graph of the objective function V is shown in figure 6. The minimum point of the surface for each η value is marked by a bold dot and defines the optimal value of the colatitude β. It can be noted that the the objective function V tends to infinite in correspondence of the boundary of the feasible domain.

Finally, we determine the optimal dimensionless rise $\tilde{f} \equiv f/L$, where $f/L = (1 - \cos\beta)/(2\sin\beta)$.

Figure 7 shows the optimal dimensionless rise $\tilde{f}$ versus λ, for different values of η. Accordingly the following design procedure can be proposed: fixed the section slenderness $\bar{w}$, the material properties, λ and f_d, and the span L, first the parameters λ and η are obtained, then the optimal value of the dimensionless rise is determined as $\tilde{f}_{opt} = \tilde{f}(\lambda, \eta)$. By observing figure 7, it emerges that the optimal values of the dimensionless rise are rather low, leading to drop arches as optimal solutions. In arch bridges the right rise-span proportion often represents a crucial aspect, that may influence the feasibility of the structure. The lower rise-span ratio implies the grater thrust and axial force, making the arch particularly suitable for bearing axial force but also leading to an increasing of the substructure cost. In this framework, the proposed methodology, allowing to calculate the optimal value of the arch rise in function of span, section amd material properties, represents an effective tool for the preliminary design of arch bridges. In particular, due to the mechanical and geometrical hypotheses at the base of the method, it is mainly suitable for the design of steel arch bridges [19].

5
CONCLUSIONS

In the present study an analytical solution for the optimal shape of a plane end-clamped arch subjected to its self weight and to a uniformly distributed vertical load has been presented. The arch volume, representative of the material cost, has been set as objective function. Optimal solutions have been derived by assuming that the normal stress reaches its maximum absolute value at the clamped end sections. Some simple rules for predesign and sensitivity scopes have finally been proposed in a dimensionless form.

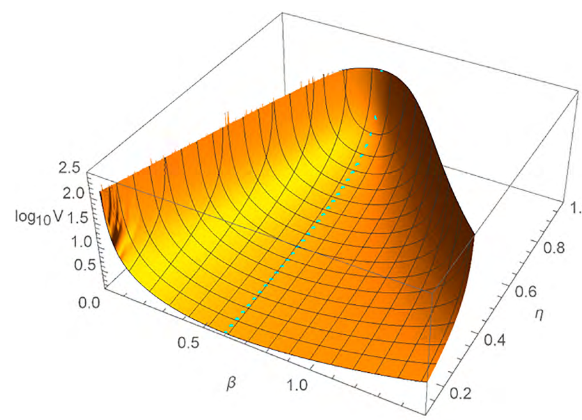

Figure 6. Objective function V versus β and η.

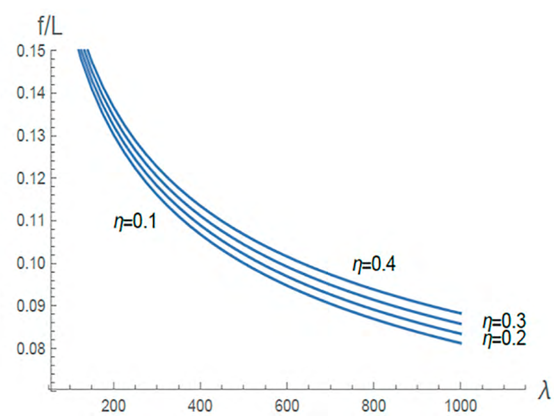

Figure 7. Optimal dimensionless rise $\tilde{f}_{opt}$ versus λ for different values of η.

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